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TWO TREE-WIDTH-LIKE GRAPH INVARIANTS

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In this paper we introduce two tree-width-like graph invariants. The first graph invariant, which we denote by $\nu^{=}(G)$, is defined in terms of positive semi-definite matrices and is similar to the graph invariant $\nu(G)$, introduced by Colin de Verdière in [J. Comb. Theory, Ser. B., 74:121–146, 1998]. The second graph invariant, which we denote by $\theta(G)$, is defined in terms of a certain connected subgraph property and is similar to $\lambda(G)$, introduced by van der Holst, Laurent, and Schrijver in [J. Comb. Theory, Ser. B., 65:291–304, 1995]. We give some theorems on the behaviour of these invariants under certain transformations. We show that $\nu^{=}(G)=\theta(G)$ for any graph G with $\nu^{=}(G)\leq 4$, and we give minimal forbidden minor characterizations for the graphs satisfying $\nu^{=}(G)\leq k$ for $k=1,2,3,4$.

1. Introduction

In [5] Colin de Verdière introduced the graph parameter $\nu(G)$, whose definition is similar to the definition of the graph parameter $\mu(G)$ (see [3]). For any graph $G=(V,E)$ with vertex set $V=\{1,\dots,n\}$, $\mu(G)$ is defined as the largest corank of any symmetric $n\times n$ matrix $M=(m_{i,j})$ with $m_{i,j}<0$ if ij is an edge, with $m_{i,j}=0$ if $i\neq j$ and ij is not an edge, and with $m_{i,i}\in\mathbb{R}$ for $i=1,\dots,n$, such that M has exactly one negative eigenvalue and such that it fulfils a certain transversality property. The parameter $\nu(G)$ is defined for any simple graph $G=(V,E)$ with vertex set $V=\{1,\dots,n\}$ as the largest

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corank of any positive semi-definite matrix $M=(m_{i,j})$ with $m_{i,j}\neq 0$ if $i=j$ is an edge, $m_{i,j}=0$ if $i\neq j$ and ij is not an edge, and $m_{i,i}\in\mathbb{R}$ for $i=1,\ldots,n$, such that M fulfils a certain transversality property.

One of the important properties of $\mu(G)$ is the characterization of planar graphs as those graphs G with $\mu(G)\leq 3$ [3]. Despite the similar definition, $\nu(G)$ is not bounded on the class of planar graphs. On the contrary, $\nu(G)$ can be arbitrary large when G is taken from the set of all squared grids.

In this paper we study a graph invariant $\nu^{=}(G)$ (see [8]), which is analogous to $\nu(G)$ and which is defined as the largest corank of any positive semi-definite $n\times n$ matrix $M=(m_{i,j})$ with $m_{i,j}=0$ if $i\neq j$ and ij is not an edge, such that M fulfils a certain transversality property. Note the difference, where $\nu(G)$ has in its definition that $m_{i,j}\neq 0$ if ij is an edge, we have have dropped this condition. Clearly, $\nu^{=}(G)\geq\nu(G)$.

Since $\nu^{=}(G)$ can be arbitrary large when G is taken from the set of all squared grids, and since $\nu^{=}(G)$ is bounded by the tree-width of the graph G (see Section 5), the work of Robertson and Seymour (see [16]) says that either G has a large $n\times n$ grid or $\nu^{=}(G)$ is bounded from above by some number t depending on n.

In [9] van der Holst, Laurent, and Schrijver introduced a to $\mu(G)$ related graph invariant, called $\lambda(G)$. Each nonzero vector in the kernel of a matrix M used in the definition of $\mu(G)$ satisfies a certain property (see [6] for the lemma describing this property). This property was used to define the invariant $\lambda(G)$. It turns out that $\nu^{=}(G)$ as well as $\nu(G)$ also have a kind of property like this. This will be discussed in Section 3; this is an important property because it allows us to get upper bounds on $\nu^{=}(G)$. Extracting this property gives the graph invariant $\theta(G)$.

The outline of the paper is as follows. In Section 2 we recall some notions of graph theory, especially the notion of tree-width of a graph. We recall some notions of matrix theory, and we will discuss the differential structure that can be put on the set of matrices, and we say something about the transversality property. In Section 3 we give the definition of $\nu^{=}(G)$ and give some theorems about this graph invariant. In Section 4 we show that $\nu^{=}(G)$ is invariant under certain transformations if $\nu^{=}(G)$ is large enough. In Section 5 we introduce the graph invariant $\theta(G)$. In Section 6 we characterize the class of graphs G with $\nu^{=}(G)\leq t$ for $t=1,2,3$, in terms of minimal forbidden minors.

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2. Graph theory and Matrix theory

In this paper all graphs are assumed to be simple.

Let $G=(V,E)$ be a graph and let $X\subseteq V$. The subgraph of G induced by the vertices of X is denoted by $G[X]$.

Let $G=(V,E)$ be a graph. If e is an edge of $G=(V,E)$ then the graph obtained from G by *deleting* edge e is the graph $(V,E\setminus\{e\})$ and is denoted by $G\setminus e$. If e is not a loop then G/e denotes the graph obtained from G by *contracting* edge e, that is, by deleting e and identifying the ends of e. A *minor* of G is a graph that can be obtained from G by a series of deletions and contractions of edges and deletions of isolated vertices from G. A graph G has an H-minor if it has a minor isomorphic to H. A minor H of G with $H\neq G$ is called a *proper minor*. A class $\mathcal{G}$ of graphs is closed under taking minors and under taking isomorphisms, if for every graph G in $\mathcal{G}$, each minor belongs to $\mathcal{G}$, and if any graph isomorphic to G belongs to $\mathcal{G}$. A graph H is a *forbidden minor* for $\mathcal{G}$ if H does not belong to $\mathcal{G}$. If additionally each proper minor belongs to $\mathcal{G}$, then H is called a *minimal forbidden minor*. A theorem of Robertson and Seymour [14] says that for any class of graph $\mathcal{G}$ closed under taking minors and isomorphisms, the collection of all minimal forbidden minors for $\mathcal{G}$ is finite (up to isomorphism).

A graph invariant $f(G)$ is *monotone under taking minors* if $f(G')\leq f(G)$ for every minor G' of G. The class of graphs G such that $f(G)\leq k$ for some integer k is closed under taking minors (and under taking isomorphism), so, by the theorem of Robertson and Seymour, it can be described in terms of a finite collection of minimal forbidden minors.

Tree-width. For any graph $G=(V,E)$, a *tree-decomposition* of G is a pair (X,T), where T is a tree and X is a family $\{X_t \mid t\in V(T)\}$ of subsets of V with the following properties.

(1) $\bigcup_{t\in V(T)} X_t=V$.
(2) For every $e=uw\in E$ there is a $t\in V(T)$ such that $u,w\in X_t$.
(3) For every $v\in V$, the set $\{t\in V(T)|v\in X_t\}$ induces a connected subgraph in T.

The width of the tree-decomposition is $\max\{|X_t|-1 \mid t\in V(T)\}$. The *tree-width* of G is the minimum width of any tree-decomposition of G; see [15]. The tree-width of a graph G is denoted by $\operatorname{tw}(G)$.

Equivalently, tree-width can be defined as follows. For a fixed positive integer k, a k-tree is defined as follows. The complete graph on k vertices, K_k, is a k-tree and every k-tree with $n>k$ vertices can be constructed from a k-tree with $n-1$ vertices by adding to it a vertex adjacent to all vertices of a subgraph isomorphic to K_k. A graph that is a subgraph of a k-tree is called a *partial k-tree*. The tree-width of a graph G is the minimum k such that G is a partial k-tree.

Note that the tree-width of a graph is monotone under taking minors; that is, if H is a minor of G then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.

For tree-width there are the following forbidden minors theorems.

Theorem 2.1. $\mathrm{tw}(G) \leq 1$ *if and only if* G *has no* K_3*-minor.*

Theorem 2.2. $\mathrm{tw}(G) \leq 2$ *if and only if* G *is a series-parallel graph; that is, if and only if* G *has no* K_4*-minor.*

Theorem 2.3. *[1]* $\mathrm{tw}(G) \leq 3$ *if and only if* G *has no* K_5*-,* V_8*-,* $K_{2,2,2}$*- or* $C_5 \times K_2$*-minor.*

Further we need the following theorem which follows from Theorem 2.3 by applying the splitter theorem [17].

Theorem 2.4. *Each graph* G *without* K_5*- or* $K_{2,2,2}$*-minor can be obtained by taking* (≤ 2)*-sums of graphs with tree-width* ≤ 3*, of* V_8 *and of* $C_5 \times K_2$*.*

Matrix theory. For matrix theory we refer to [13,18]. If $M = (m_{i,j})$ is a matrix, then M^{T} denotes the transpose of M. The transpose of M is the matrix $N = (n_{i,j})$ with $n_{i,j} = m_{j,i}$ for all entries of N. A matrix M is symmetric if $M^{\mathrm{T}} = M$.

If M is a symmetric matrix, then all eigenvalues of M are real numbers, and hence we can order the eigenvalues of M and thus we can speak about the smallest, second but smallest, etc. A symmetric matrix M is positive semi-definite if the smallest eigenvalue ≥ 0. It is positive definite if the smallest eigenvalue > 0. For symmetric matrices there is the following characterization of positive definite and positive semi-definite matrices. A symmetric matrix M is positive definite if and only if $x^{\mathrm{T}} M x > 0$ for all $x \neq 0$. A symmetric $n \times n$ matrix M is positive semi-definite if and only if $x^{\mathrm{T}} M x \geq 0$ for all $x \in \mathbb{R}^n$. If, moreover $x \neq 0$ and $x^{\mathrm{T}} M x = 0$, then $x \in \ker(M)$.

Let M be a symmetric matrix. *Sylvester's Law of Inertia* says that $A^{\mathrm{T}} M A$ has the same number of negative, the same number of zero, and the same number of positive eigenvalues as M whenever A is a nonsingular matrix; that is, whenever the kernel of A only contains the all-zero vector.

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$. Let $M = (m_{i,j})$ be an symmetric $n \times n$ matrix. If C_1 and C_2 are subsets of V, then by $M_{C_1 \times C_2}$ we denote the submatrix of M consisting of all entries $m_{i,j}$ with $i \in C_1$ and $j \in C_2$. Let $x \in \mathbb{R}^n$ and let $S \subseteq V$. By x_S we denote the subvector of x consisting of all entries x_i with $i \in S$.

If $x \in \mathbb{R}^n$, then the support of x is denoted by $\text{supp}(x)$. So $\text{supp}(x) := \{i \mid x_i \neq 0\}$.

Proposition 2.5. *Let M be a positive semi-definite matrix. Let $x \in \ker(M)$ and let $S := \text{supp}(x)$. Then $M_{S\times S}$ is singular. Conversely, if $M_{S\times S}$ is singular, then there is an $x \in \ker(M)$ with $\text{supp}(x) \subseteq S$.*

Differential structure. Let $\mathcal{S}_n$ denote the manifold of all symmetric $n\times n$ matrices. Let $\mathcal{A}_{n,k}$ denote the set of all symmetric $n\times n$ matrices with corank k. If $\mathcal{S}$ is a manifold and $M \in \mathcal{S}$, then the tangent space of $\mathcal{S}$ at M is denoted by $T_M\mathcal{S}$. So the tangent space of $\mathcal{A}_{n,k}$ at M is denoted by $T_M\mathcal{A}_{n,k}$. We have

Theorem 2.6. *$\mathcal{A}_{n,k}$ is a smooth submanifold of $\mathcal{S}_n$. The tangent space of $\mathcal{A}_{n,k}$ at M is equal to the set of all symmetric matrices C with $x^{\mathsf{T}}Cx = 0$ for all $x \in \ker(M)$.*

This also shows that $\mathcal{A}_{n,k}$ has codimension $\frac{1}{2}k(k+1)$.

It should be noted that, in general, the set of all symmetric $n\times n$ matrices N with corank *at least* k, $k \geq 1$, does not form a smooth submanifold of the manifold of symmetric matrices.

The space $\mathcal{S}_n$ can be made an orthogonal space by introducing the inner product $\langle C, D\rangle = Tr(CD)$. (Note that $Tr(CD) = Tr(DC)$ is true for any $n \times m$ matrix C and any $n \times m$ matrix D.) With this inner product, the normal space of any linear subspace is defined.

The normal space of a submanifold in a manifold at some element p is the space of all vectors in the tangent space of the manifold at p orthogonal to the tangent space of the submanifold at p. The normal space of $\mathcal{A}_{n,k}$ at M in $\mathcal{S}_n$, which we denote by $N_M\mathcal{A}_{n,k}$, is the space generated by all matrices of the form xx^{T} with $x \in \ker(M)$. This is easy to see: $x^{\mathsf{T}}Cx = 0$ if and only if $Tr(Cxx^{\mathsf{T}}) = 0$, so the matrices xx^{T} belong to $N_M\mathcal{A}_{n,k}$. A dimension argument then shows that the space spanned by all matrices xx^{T} with $x \in \ker(M)$ is the normal space of $\mathcal{A}_{n,k}$ at M. Using the fact that each column of a symmetric $n \times n$ matrix X with $MX = 0$ belongs to $\ker(M)$, one can show that the normal space of $\mathcal{A}_{n,k}$ at M is equal to the space of all symmetric $n \times n$ matrices X with $MX = 0$.

Transversality. Let G be a graph with vertex set $\{1, \ldots, n\}$. Let $\mathcal{M}_G^{\equiv}$ denote the set of all symmetric $n \times n$ matrices $M = (m_{i,j})$ satisfying $m_{i,j} = 0$ if i and j are nonadjacent. A matrix $M \in \mathcal{M}_G^{\equiv}$ of corank k is said to fulfil the *Strong Arnol'd Property* or SAP for short if $\mathcal{A}_{n,k}$ intersects $\mathcal{M}_G^{\equiv}$ transversally at M; that is, if the linear span of the tangent space of $\mathcal{M}_G^{\equiv}$ at M and the tangent space of $\mathcal{A}_{n,k}$ at M is the space of symmetric $n \times n$ matrices.

A criterion to check if a matrix fulfils the SAP is

Criterion 1. [3,4] $M \in \mathcal{M}_G^{=}$ fulfils the SAP if and only if for each symmetric $n \times n$ matrix A there is a matrix $B \in T_M \mathcal{M}_G^{=}$ such that $x^{\mathrm{T}} A x = x^{\mathrm{T}} B x$ for each $x \in \ker(M)$.

Proof. Suppose M fulfils the SAP; that is, the linear span of $T_M \mathcal{M}_G^{=}$ and $T_M \mathcal{A}_{n,k}$ is the space of all symmetric $n \times n$ matrices. Let A be any symmetric $n \times n$ matrix. Then A belongs to the linear span of $T_M \mathcal{M}_G^{=}$ and $T_M \mathcal{A}_{n,k}$, and hence $A = B + C$ with $B \in T_M \mathcal{M}_G^{=}$ and $C \in T_M \mathcal{A}_{n,k}$. By Theorem 2.6 the tangent space of $\mathcal{A}_{n,k}$ at matrix M is the space of all matrices C with $x^{\mathrm{T}} C x = 0$ for each $x \in \ker(M)$. Hence $x^{\mathrm{T}} A x = x^{\mathrm{T}} B x$ for each $x \in \ker(M)$.

For the converse, let A be any symmetric $n \times n$ matrix. Then there is a matrix $B \in T_M \mathcal{M}_G^{=}$ such that $x^{\mathrm{T}} A x = x^{\mathrm{T}} B x$ for all $x \in \ker(M)$. So $x^{\mathrm{T}}(A - B)x = 0$ for all $x \in \ker(M)$, which implies that $A - B \in T_M \mathcal{A}_{n,k}$. ∎

Another criterion for a matrix to fulfil the SAP is the following.

Criterion 2. [10,12] M fulfils the SAP if and only if there is no nonzero symmetric matrix $X = (x_{i,j})$ with $x_{i,i} = 0$ and $x_{i,j} = 0$ if i and j are adjacent in G, such that $MX = 0$.

The relation between Criterion 1 and Criterion 2 is the following. Let $y_1, \ldots, y_k$ be a basis of $\ker(M)$ and let $Y := (y_1, \ldots, y_k)$ (this is the $n \times k$ matrix with the ith column equal to y_i). Then by Criterion 1, M does not fulfil the SAP if and only if the space of all matrices $Y^{\mathrm{T}} A Y$, $A \in T_M \mathcal{M}_G^{=}$, is not the whole space of all symmetric $k \times k$ matrices. This means that there exists a symmetric $k \times k$ matrix C, orthogonal to the space of all matrices $Y^{\mathrm{T}} A Y$, for $A \in T_M \mathcal{M}_G^{=}$; that is, $Tr(Y^{\mathrm{T}} A Y C) = 0$ for all $A \in T_M \mathcal{M}_G^{=}$. Since $Tr(Y^{\mathrm{T}} A Y C) = Tr(A Y C Y^{\mathrm{T}}) = 0$, we see that $Y C Y^{\mathrm{T}}$ is orthogonal to $T_M \mathcal{M}_G^{=}$. So $X = (x_{i,j}) := Y C Y^{\mathrm{T}}$ is a matrix with $x_{i,i} = 0$ and $x_{i,j} = 0$ if i and j are adjacent in G, for which $MX = 0$.

3. Definition of $\nu^{=}(G)$

The graph parameter $\nu^{=}(G)$ is defined as the largest corank of any positive semi-definite matrix $M \in \mathcal{M}_G^{=}$ fulfilling the SAP. By definition, $\nu(G) \leq \nu^{=}(G)$.

The same way as it is proved that $\nu(G') \leq \nu(G)$ if G' is a minor of G [5,7,11] it can be shown that:

Theorem 3.1. *If G' is a minor of G then $\nu^{=}(G') \leq \nu^{=}(G)$.*

Hence for any fixed n, the class of graphs G with $\nu^{=}(G) \leq n$ is closed under taking minors and under taking isomorphisms, implying that there is a finite collection of minimal forbidden minors. In Section 6, we shall give the minimal forbidden minors for the class of graphs G with $\nu^{=}(G) \leq t$, for $t = 1, 2, 3, 4$.

Proposition 3.2. *$\nu^{=}(K_n) = n$.*

Proof. Let M be the all-zero matrix. Then M is positive semi-definite and has corank n. By Criterion 2, M fulfils the SAP. Therefore $\nu^{=}(K_n) \geq n$. Clearly, $\nu^{=}(K_n) \leq n$, hence $\nu^{=}(K_n) = n$. ∎

Since for each proper minor G of K_n, $\nu^{=}(G) < n$, we see that K_n is a minimal forbidden minor for the class of graphs G with $\nu^{=}(G) \leq n-1$, for $n > 0$.

The next proposition gives us a tool to find upper bounds for $\nu^{=}(G)$.

Proposition 3.3. *Let $M \in \mathcal{M}_G^{=}$ be positive semi-definite and suppose M fulfils the SAP. Then for each $x \in \ker(M)$ the support of x induces a connected subgraph of G.*

Proof. Let $x \in \ker(M)$. Suppose to the contrary that $G[\operatorname{supp} x]$ is disconnected. Let C_1 be a component of $G[\operatorname{supp} x]$, let $C_2 := G[\operatorname{supp} x] - V(C_1)$, and let $C := G - (V(C_1) \cup V(C_2))$. Let $y \in \mathbb{R}^V$ with $y_i = x_i$ if $i \in V(C_1)$ and $y_i = 0$ if $i \notin V(C_1)$. Let $z \in \mathbb{R}^V$ with $z_i = x_i$ if $i \in V(C_2)$ and $z_i = 0$ if $i \notin V(C_2)$. Then $y^{\mathrm{T}} M y + z^{\mathrm{T}} M z = x^{\mathrm{T}} M x = 0$, as $y^{\mathrm{T}} M z = 0$. Since M is positive semi-definite, $y^{\mathrm{T}} M y \geq 0$ and $z^{\mathrm{T}} M z \geq 0$, and hence $y^{\mathrm{T}} M y = 0$ and $z^{\mathrm{T}} M z = 0$. This means that y and z also belong to $\ker(M)$. Let $X = (x_{i,j}) := yz^{\mathrm{T}} + zy^{\mathrm{T}}$. Then $x_{i,j} = 0$ if $i = j$ or if ij is an edge of G. But $MX = 0$, showing that M does not fulfil the SAP. ∎

A simple translation, using Proposition 2.5, gives

Corollary 3.3a. *Let $M \in \mathcal{M}_G^{=}$ be positive semi-definite and suppose M fulfils the SAP. Let V_1 and V_2 be disjoint subsets of $V(G)$ such that there is no edge connecting a vertex of V_1 with a vertex of V_2 in G. Then $M_{V_1 \times V_1}$ or $M_{V_2 \times V_2}$ is positive definite.*

In [8] it is shown that, if G has a k-connected minor, then $\nu(G) \geq k$. This implies:

Theorem 3.4. *If G contains a k-connected minor then $\nu^{=}(G) \geq k$.*

We will see that for $k=1,\dots,5$, $\nu^{=}(G)\geq k$ implies that G has a $(k-1)$-connected minor. Of course this cannot be true in general because every planar graphs G always has a (≤ 5)-vertex cut, while $\nu^{=}(G)$ can be arbitrarily large in the class of planar graphs, as is shown by Colin de Verdière for $\nu(G)$ [5].

The *suspension* of a graph $G=(V,E)$ is the graph obtained from G by adding a new vertex s and connecting this vertex to each vertex of G by one edge; the vertex s is called *the suspended vertex*.

Theorem 3.5. *Let $S(G)$ denote the suspension of G. Then $\nu^{=}(S(G)) = \nu^{=}(G)+1$.*

Proof. We first show that $\nu^{=}(S(G))\geq\nu^{=}(G)+1$. Let $M\in\mathcal{M}^{=}_{G}$ be positive semi-definite, with corank $\nu^{=}(G)$ and fulfilling the SAP. Let

$$M' := \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}.$$

Then M' belongs to $\mathcal{M}^{=}_{S(G)}$, is positive semi-definite, and has corank $\nu^{=}(G)+1$. If M' does not fulfil the SAP, then there is a nonzero symmetric matrix $X'=(x'_{i,j})$ with $x'_{i,j}=0$ if $i=j$ or ij is an edge of $S(G)$, such that $M'X'=0$. We can write

$$X' = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix},$$

where $X=(x_{i,j})$ has $x_{i,j}=0$ if $i=j$ or ij is an edge of G. Since $M'X' = MX=0$ and X' is nonzero, X is nonzero, and hence M does not fulfil the SAP – a contradiction.

Next we show that $\nu^{=}(S(G))\leq\nu^{=}(G)+1$. Let $M'\in\mathcal{M}^{=}_{G}$ be positive semi-definite, with corank $\nu^{=}(S(G))$, and fulfilling the SAP. Let s be the suspended vertex. Deleting the sth row and column from M' gives a matrix $M\in\mathcal{M}^{=}_{G}$. Since M is a principal submatrix of M', M is positive semi-definite and belongs to $\mathcal{M}^{=}_{G}$. If M does not fulfil the SAP, then there is a nonzero symmetric matrix $X=(x_{i,j})$ with $x_{i,j}=0$ if $i=j$ or ij is an edge of G, such that $MX=0$. Let

$$X' = (x'_{i,j}) := \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}.$$

Then X' is nonzero, $x'_{i,j}=0$ if $i=j$ or ij is an edge of $S(G)$, and $M'X' = 0$. Hence M' does not fulfil the SAP – a contradiction. If M has corank $\nu^{=}(S(G))$, then, since M fulfils the SAP,

$$K := \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$$

is a matrix in $\mathcal{M}^{=}_{S(G)}$ showing that M' does not have the largest corank. Hence $\nu^{=}(S(G))=\nu^{=}(G)+1$. ∎

4. Some invariant transformations

We now give some transformation on graphs under which $\nu^{=}(G)$ is invariant.

A graph $G=(V,E)$ is a *clique sum* of the graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ if V_1 and V_2 are subsets of V such that $V_1\cap V_2$ induces a clique in G_1 and G_2 and if $E=E_1\cup E_2$.

Theorem 4.1. *Let G be a clique sum of G_1 and G_2. Then* $\nu^{=}(G)=\max\{\nu^{=}(G_1),\nu^{=}(G_2)\}$.

Proof. Since $\nu^{=}$ is monotone under taking minors,

$$\nu^{=}(G)\geq \max\{\nu^{=}(G_1),\nu^{=}(G_2)\}. \tag{1}$$

To see equality, let $S:=VG_1\cap VG_2$. Let $M\in\mathcal{M}^{=}_{G}$ be positive semi-definite, with $\operatorname{corank}(M)=\nu^{=}(G)$, and fulfilling the SAP. Suppose $\nu^{=}(G)>\max\{\nu^{=}(G_1),\nu^{=}(G_2)\}$. Then there is a nonzero $x\in\ker(M)$ with $x_S=0$. Let $C_1:=VG_1\setminus S$ and $C_2:=VG_2\setminus S$. By Corollary 3.3a, $M_{C_1\times C_1}$ or $M_{C_2\times C_2}$ is positive definite. We assume without loss of generality that $M_{C_1\times C_1}$ is positive definite.

Write

$$M:=\begin{pmatrix} M_{C_1\times C_1} & M_{C_1\times S} & 0\\ M_{S\times C_1} & M_{S\times S} & M_{S\times C_2}\\ 0 & M_{C_2\times S} & M_{C_2\times C_2}\end{pmatrix}.$$

So $M_{S\times C_1}=M^{\mathrm{T}}_{C_1\times S}$ and $M_{S\times C_2}=M^{\mathrm{T}}_{C_2\times S}$.

Let

$$A:=\begin{pmatrix} I & -M^{-1}_{C_1\times C_1}M_{C_1\times S} & 0\\ 0 & I & 0\\ 0 & 0 & I\end{pmatrix}.$$

Then, by Sylvester's law of inertia, the matrix

$$A^{\mathrm{T}}MA=\begin{pmatrix} M_{C_1\times C_1} & 0 & 0\\ 0 & M_{S\times S}-M_{S\times C_1}M^{-1}_{C_1\times C_1}M_{C_1\times S} & M_{S\times C_2}\\ 0 & M_{C_2\times S} & M_{C_2\times C_2}\end{pmatrix}$$

has the same number of negative, the same number of zero, and the same number of positive eigenvalues as M. This means that the matrix

$$M' := \begin{pmatrix} M_{S\times S} - M_{S\times C_1} M^{-1}_{C_1\times C_1} M_{C_1\times S} & M_{S\times C_2} \\ M_{C_2\times S} & M_{C_2\times C_2} \end{pmatrix}$$

is positive semi-definite and that it has corank equal to $\nu^{=}(G)$. Since S induces a clique in G_2, $M' \in \mathcal{M}^{=}_{\overline{G}_2}$. If we have shown that M' fulfils the SAP, then $\nu^{=}(G_2) = \nu^{=}(G)$, which contradicts the assumption, and thus proving equality in (1).

Let $X' = x'_{i,j}$ be a symmetric matrix with $M'X' = 0$ and $x'_{i,j} = 0$ if i and j are adjacent or $i = j$. We write

$$X' = \begin{pmatrix} 0 & X'_{S\times C_2} \\ X'_{C_2\times S} & X'_{C_2\times C_2} \end{pmatrix},$$

where ${X'_{S\times C_2}}^{\mathrm{T}} = X'_{C_2\times S}$. Let $Z := -M^{-1}_{C_1\times C_1} M_{C_1\times S} X'_{S\times C_2}$ and

$$X = (x_{i,j}) := \begin{pmatrix} 0 & 0 & Z \\ 0 & 0 & X'_{S\times C_2} \\ Z^{\mathrm{T}} & X'_{C_2\times S} & X'_{C_2\times C_2} \end{pmatrix}.$$

Then X is a symmetric matrix with $x_{i,j} = 0$ if i and j are adjacent or $i = j$, and $MX = 0$. This implies $X = 0$ and hence $X' = 0$. Hence M' fulfils the SAP. ∎

From this theorem we get the following corollaries.

Corollary 4.1a. *Let $G = (V, E)$ be a graph and v be a vertex of G with exactly two neighbors. Let $G' = (V', E')$ be obtained from G by deleting v (and incident edges) and connecting the two neighbors of v by an edge. If $\nu^{=}(G') \geq 3$ then $\nu^{=}(G) = \nu^{=}(G')$.*

Proof. Note that G is a subgraph of a clique sum of G' with K_3. By Theorem 4.1 and Theorem 3.1,

$$\nu^{=}(G) \leq \max\{\nu^{=}(G'), \nu^{=}(K_3)\}.$$

Hence $\nu^{=}(G) \leq \nu^{=}(G')$, since $\nu^{=}(G') \geq 3 = \nu^{=}(K_3)$. By Theorem 3.1, $\nu^{=}(G') = \nu^{=}(G)$. ∎

Hence, if $\nu^{=}(G') \geq 3$, then each subdivision G of G' has $\nu^{=}(G) = \nu^{=}(G')$.

A graph G' is obtained from G by a *$Y\Delta$-transformation* if there is a vertex v of G of degree 3, such that G' is obtained from G by deleting the vertex v and its incident edges, and by adding an edge between each pair of vertices of the neighbourhood of v. A graph G' is obtained from G by a *ΔY-transformation* if G' can be obtained by deleting the edges of a triangle of G and by adding a new vertex and edges from this vertex to all vertices of the triangle.

Corollary 4.1b. *Let G be a graph and let G' be obtained from G by applying a $Y\Delta$-transformation. If $\nu^{=}(G') \geq 4$ then $\nu^{=}(G') \geq \nu^{=}(G)$.*

Proof. G is a subgraph of a clique sum of G' and K_4. Hence by Theorem 4.1 and Theorem 3.1,

$$\nu^{=}(G) \leq \max\{\nu^{=}(G'), \nu^{=}(K_4)\}.$$

Hence $\nu^{=}(G) \leq \nu^{=}(G')$, since $\nu^{=}(G') \geq 4 = \nu^{=}(K_4)$. ∎

In [2] Bacher and Colin de Verdière proved the similar statement of Corollary 4.1b for the graph parameter $\mu(G)$, and proved moreover that $\mu(G') \geq \mu(G)$ if G' is obtained from G by a ΔY-transformation. Regarding $\nu^{=}(G)$, there are graphs G for which $\nu^{=}(G') \geq \nu^{=}(G)$ is not true if G' is obtained from G by applying a ΔY-transformation. For example, applying a ΔY-transformation on K_n gives us a graph that is a subgraph of clique sum of K_{n-1}'s, but $\nu^{=}(K_n) = n$, whereas a clique sum H of K_{n-1}'s has $\nu^{=}(H) = n-1$.

5. An invariant based on the connected support

Proposition 3.3 suggests to define another graph parameter, which we denote by $\theta(G)$. The parameter $\theta(G)$, defined for any graph $G = (V, E)$, is the largest $d \in \mathbb{N}$ for which there exists a d-dimensional subspace X of $\mathbb{C}^V$ such that:

(2) for each nonzero $x \in X$, $G[\mathrm{supp}(x)]$ is connected.

An equivalent characterization of $\theta(G)$ is the following. Let $G = (V, E)$ be a graph and $d \in \mathbb{N}$. Call a function $\phi : V \longrightarrow \mathbb{C}^d$ a *valid representation* if

(3) for each hyperplane H of $\mathbb{C}^d$, the set $\phi^{-1}(\mathbb{C}^d \setminus H)$ is nonempty and induces a connected subgraph of G.

(A subset H of $\mathbb{C}^d$ is called a *hyperplane* if $H=\{x\in\mathbb{C}^d \mid c^{\mathrm{T}}x=0\}$ for some $c\in\mathbb{C}^d$.) Note that if $\phi:V\longrightarrow\mathbb{C}^d$ is a valid representation, then the vectors $\phi(v)$ $(v\in V)$ span $\mathbb{C}^d$ (since otherwise there would exists a hyperplane H with $\phi(v)\in H$ $(v\in V)$, which implies that $\phi^{-1}(\mathbb{C}^d\setminus H)=\emptyset$).

That $\theta(G)$ is equal to the largest d for which there is a valid representation $\phi:V\longrightarrow\mathbb{C}^d$ is easy to see. Suppose X is a d-dimensional subspace of $\mathbb{C}^V$ satisfying (2). Let $x_1,\dots,x_d$ form a basis of X. Define $\phi(v):=(x_1(v),\dots,x_d(v))$ for each $v\in V$. Then $\phi:V\longrightarrow\mathbb{C}^d$ is a valid representation.

Conversely, let $\phi:V\longrightarrow\mathbb{C}^d$ be a valid representation. Define for any $c\in\mathbb{C}^d$ the $x_c\in\mathbb{C}^V$ by: $x_c(v):=c^{\mathrm{T}}\phi(v)$ for $v\in V$. Then $X:=\{x_c\mid c\in\mathbb{C}^d\}$ satisfies (2).

The following theorem shows that θ is an invariant that is monotone under taking minors.

Theorem 5.1. *If G' is a minor of G then $\theta(G')\leq\theta(G)$.*

Proof. If G' arises from G by deleting an isolated vertex v_0, the inequality $\theta(G')\leq\theta(G)$ is easy: if $\phi:V(G')\longrightarrow\mathbb{C}^d$ is a valid representation for G' with $d=\theta(G')$, then defining $\phi(v_0):=0$ gives a valid representation for G.

So we may assume that $G'=(V',E')$ arises from $G=(V,E)$ by deleting or contracting one edge $e=uw$. Let $\phi':V'\longrightarrow\mathbb{C}^d$ be a valid representation for G' with $d=\theta(G')$. If G' arises from G by deleting e, then $V=V'$, and ϕ' is also a valid representation for G. Hence $\theta(G)\geq d=\theta(G')$.

If G' arises from G by contracting e, let v_0 be the vertex of G' which arised by contracting e. Define $\phi(u):=\phi(w):=\phi'(v_0)$, and define $\phi(v):=\phi'(v)$ for all other vertices v of G. Then ϕ is a valid representation of G. ∎

This theorem says that for each $t\in\mathbb{N}$ the class of graphs G satisfying $\theta(G)\leq t$ is closed under taking minors. Hence there is a finite set of minimal forbidden minors for any such class of graphs. That is, for each $t\in\mathbb{N}$ there is a finite collection $\mathcal{C}_t$ of graphs such that $\theta(G)>t$ if and only if G has at least one of the graphs in $\mathcal{C}_t$ as a minor.

Theorem 5.2. $\theta(K_n)=n$.

Proof. Let $X:=\mathbb{C}^V$. Then each $x\in X$ has $G[\mathrm{supp}(x)]$ connected. Hence $\theta(K_n)\geq n$.

On the other hand, it is clear that $\theta(K_n)\leq n$. ∎

It is clear that $G=K_n$ is the only graph with n vertices satisfying $\theta(G)=n$. Hence each proper minor G' of K_n satisfies $\theta(G')<n$. So for each $n\geq 0$, $K_{n+1}\in\mathcal{C}_n$.

Theorem 5.3. *If $H = G \setminus \{v\}$, then $\theta(G) \leq \theta(H) + 1$.*

This implies that deleting or contracting an edge uw of G decreases $\theta(G)$ by at most 1, as the graph with uw deleted or contracted has $G \setminus \{w\}$ as a subgraph.

Since for each positive semi-definite matrix $M \in \mathcal{M}_G^=$ fulfilling the SAP, each nonzero vector $x \in \ker(M)$ has $G[\mathrm{supp}(x)]$ nonempty and connected, we immediately have

$$\nu^=(G) \leq \theta(G). \tag{4}$$

We will now study the behaviour of $\theta(G)$ under clique sums. We begin with an easy observation. If $S \subseteq V$ then we let $\phi(S)$ be the linear span of $\{\phi(s) \mid s \in S\}$.

Observation 1. *If $\phi : V \longrightarrow \mathbb{C}^d$ is a valid representation, then for each $S \subset V$ there is at most one component C of $G \setminus S$ for which $\phi(V(C))$ is not contained in $\phi(S)$.*

Theorem 5.4. *If $G = (V, E)$ is clique sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ then*

$$\theta(G) = \max\{\theta(G_1), \theta(G_2)\}. \tag{5}$$

Proof. Since θ is monotone under taking minors,

$$\theta(G) \geq \max\{\theta(G_1), \theta(G_2)\}.$$

To see equality, let $T := V_1 \cap V_2$. Let $\phi : V \longrightarrow \mathbb{C}^d$ be a valid representation for G with $d = \theta(G)$. Then $\phi(V_1) \subseteq \phi(T)$ or $\phi(V_2) \subseteq \phi(T)$; we assume that $\phi(V_1) \subseteq \phi(T)$. Let $\psi : V_2 \longrightarrow \mathbb{C}^d$ be defined by $\psi(v) := \phi(v)$ for $v \in V_2$. Since $\phi(V_1) \subseteq \phi(T)$, $\psi(V_2)$ is d-dimensional. If H is a hyperplane of $\mathbb{C}^d$ for which $\psi^{-1}(\mathbb{C}^d \setminus H)$ is not connected, then, because T induces a clique in G_2 and any path from $V_1 \backslash T$ to $V_2 \backslash T$ has a vertex in T, $\phi^{-1}(\mathbb{C}^d \backslash H)$ is not connected. So $\theta(G) \leq \theta(G_2)$, and hence the validity of (5). ∎

From Theorem 5.4 we derive:

Theorem 5.5. $\theta(G) \leq \mathrm{tw}(G) + 1$.

Proof. G is a subgraph of a k-tree for $k = \mathrm{tw}(G)$. As θ is monotone under taking subgraphs, it suffices to prove the statement for k-trees. But as each k-tree can be obtained by applying clique sums on K_{k+1}, and since $\theta(K_{k+1}) = k+1$, the statement follows from Theorem 5.4. ∎

From (4) it follows:

Theorem 5.6. $\nu^=(G) \leq \theta(G) \leq \mathrm{tw}(G) + 1$.

6. Forbidden minors

Using Theorem 4.1, 5.6, 5.4, and 5.5 we can now give the forbidden minors for the class of graphs G satisfying $\nu^{=}(G) \leq t$ for $t = 1, \ldots, 4$, and the forbidden minors for the class of graphs G with $\theta(G) \leq t$ for $t = 1, \ldots, 4$. It will turn out that $\nu^{=}(G) = \theta(G)$ if $\nu^{=}(G) \leq 4$.

The first proposition is clear.

Proposition 6.1. *$\nu^{=}(G) \leq 1$ if and only if G has no edges. The same holds if $\nu^{=}(G)$ is replaced by $\theta(G)$.*

Next we have

Proposition 6.2. *$\nu^{=}(G) \leq 2$ if and only if G has tree-width ≤ 1; that is, if and only if G is a forest. The same holds if $\nu^{=}(G)$ is replaced by $\theta(G)$.*

Proof. If $\nu^{=}(G) \leq 2$ then G has no K_3-minor, as $\nu^{=}(K_3) = 3$.

Conversely, if the underlying simple graph of G is a forest, then $\operatorname{tw}(G) \leq 1$, and hence by Theorem 5.6, $\nu^{=}(G) \leq 2$. ∎

Proposition 6.3. *$\nu^{=}(G) \leq 3$ if and only if G has tree-width ≤ 2; that is, if and only if G is series-parallel. The same holds if $\nu^{=}(G)$ is replaced by $\theta(G)$.*

Proof. If $\nu^{=}(G) \leq 3$ then G has no K_4-minor, as $\nu^{=}(K_4) = 4$.

Conversely, if G is a series-parallel graph, then $\operatorname{tw}(G) \leq 2$, and hence by Theorem 5.6, $\nu^{=}(G) \leq 3$. ∎

In contrast to what the propositions given above might suggest, $\nu^{=}(G)$ is not equal to $\operatorname{tw}(G) + 1$ for $\nu^{=}(G) = 4$. However, we will see that for 3-connected graphs the only exceptions for the statement $\operatorname{tw}(G) \leq 3$ if and only if $\nu^{=}(G) \leq 4$ are the two graphs V_8 and $C_5 \times K_2$. That is, we will show that $\nu^{=}(G) \leq 4$ for a 3-connected graph G if and only if $\operatorname{tw}(G) \leq 3$ or $G = V_8$ or $G = C_5 \times K_2$. To this end, we first show:

Proposition 6.4. *$\nu^{=}(K_{2,2,2}) = \theta(K_{2,2,2}) = 5$.*

Proof. As $K_{2,2,2}$ has tree-width 4, $\nu^{=}(K_{2,2,2}) \leq \theta(K_{2,2,2}) \leq 5$.

Number the vertices of $K_{2,2,2}$ as in Figure 1. To see that $\nu^{=}(K_{2,2,2}) = 5$, and hence $\theta(K_{2,2,2}) = 5$, let

$$M := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

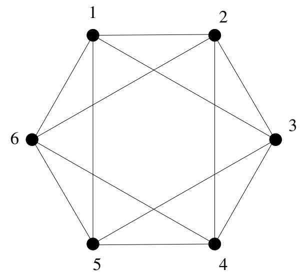

Fig. 1. Numbering of $K_{2,2,2}$

Then, clearly, $M \in \mathcal{M}^{=}_{K_{2,2,2}}$, and M is positive semi-definite and has corank equal to 5, with kernel the space spanned by the vectors $(1,0,0,0,0,0)^{\mathrm{T}}$, $(0,1,0,0,0,0)^{\mathrm{T}}$, $(0,0,0,0,0,1)^{\mathrm{T}}$, $(0,0,1,-1,0,0)^{\mathrm{T}}$ and $(0,0,1,0,-1,0)^{\mathrm{T}}$. To show that M fulfils the SAP, let $X=(x_{i,j})$ be a symmetric matrix with $x_{i,j}=0$ if i and j are adjacent or $i=j$, and $MX=0$. So

$$X := \begin{pmatrix} 0 & 0 & 0 & x_{1,4} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{2,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{3,6} \\ x_{4,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{5,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{6,3} & 0 & 0 & 0 \end{pmatrix},$$

and $MX=0$, which implies $X=0$. Hence M fulfils the SAP. ∎

By Theorem 2.4, any graph with no K_5- or $K_{2,2,2}$-minor can be obtained by (≤ 2)-sums from graphs with tree-width ≤ 3 and from V_8 and $C_5 \times K_2$. Since graphs G with tree-width ≤ 3 have $\nu^{=}(G) \leq 4$, knowing $\nu^{=}(V_8)$ and $\nu^{=}(C_5 \times K_2)$ implies that we know the minimal forbidden minors for the class of graphs G with $\nu^{=}(G) \leq 4$. This is what we will study now.

Proposition 6.5. $\nu^{=}(V_8)=\theta(V_8)=4$.

Proof. Since K_4 is a minor of V_8, $\theta(V_8) \geq \nu^{=}(V_8) \geq 4$.

Suppose that $\theta(V_8) > 4$; that is, there is a d-dimensional subspace X of $\mathbb{C}^V$ satisfying (2). Number the vertices of V_8 as in Figure 2.

We first show that there is no vector $x \in X$ with $|\mathrm{supp}(x)| = 1$. For suppose there exists one, say x, with $x_1 \neq 0$. Let $x' \in X$ be nonzero with $x'_1 = x'_2 = x'_5 = x'_8 = 0$, which exists because $\theta(V_8) > 4$. Then $G[\mathrm{supp}(x+x')]$ is disconnected.

Let $x \in X$ be nonzero with $x_3 = x_5 = x_6 = x_8 = 0$ and let $x' \in X$ be nonzero with $x'_1 = x'_4 = x'_6 = x'_7 = 0$. Since there are no vector $y \in X$ with

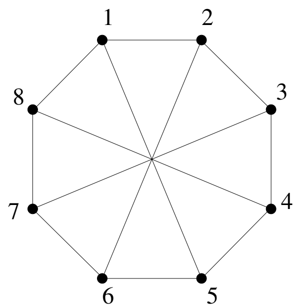

Fig. 2. Numbering of V_8

$|\text{supp}(y)|=1$, $\text{supp}(x)=\{1,2\}$ and $\text{supp}(x')=\{2,3\}$. Let $z=x'_2x-x_2x'$. Then $\text{supp}(z)=\{1,3\}$ and hence $G[\text{supp}(z)]$ is disconnected. So the assumption that $\theta(V_8)>4$ is wrong, hence $\theta(V_8)=4$ and $\nu^=(V_8)=4$. ∎

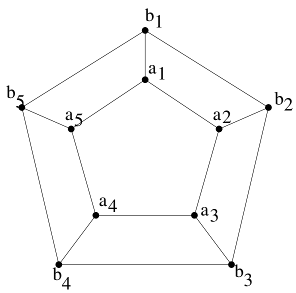

Fig. 3. Labelling of $C_5\times K_2$

Proposition 6.6. $\nu^=(C_5\times K_2)=\theta(C_5\times K_2)=4$.

Proof. Since K_4 is a minor of $C_5\times K_2$, $\theta(C_5\times K_2)\geq\nu^=(C_5\times K_2)\geq 4$.

Suppose that $\theta(C_5\times K_2)>4$; that is, there is a d-dimensional subspace X of $\mathbb{C}^V$ satisfying (2), where $d\geq 5$.

Let $a_1,\dots,a_5$, respectively $b_1,\dots,b_5$ be the vertices of one of the C_5, respectively the other C_5, and let b_i be adjacent to a_i for $i=1,\dots,5$. In what follows we assume that the indices are modulo 5. See Figure 3.

First, there is no vector $x\in X$ with $|\text{supp}(x)|=1$; the proof is similar as in the proof of Proposition 6.5.

(6) There is no vector $x\in X$ with $\text{supp}(x)=\{a_i,a_{i+1}\}$ or $\text{supp}(x)=\{b_i,b_{i+1}\}$ $(i=1,\dots,5)$.

Suppose there exists one, x say, which we may assume without loss of generality to have $\mathrm{supp}(x)=\{a_1,a_2\}$.

Let $u\in X$ be nonzero with $u_{a_1}=u_{a_4}=u_{b_2}=u_{b_5}=0$. Then $\mathrm{supp}(u)\subseteq\{a_2,a_3,b_3,b_4\}$. If $a_2\in\mathrm{supp}(u)$, let $y:=x_{a_2}u-u_{a_2}x$. Then $a_1\in\mathrm{supp}(y)$ and at least one of the elements of $\{a_3,b_3,b_4\}$ belongs to $\mathrm{supp}(y)$, and hence $G[\mathrm{supp}(y)]$ is disconnected. So $\mathrm{supp}(u)\subseteq\{a_3,b_3,b_4\}$. If $b_3\notin\mathrm{supp}(u)$ then $G[\mathrm{supp}(u)]$ is disconnected. If $a_3\notin\mathrm{supp}(u)$ then $G[\mathrm{supp}(x+u)]$ is disconnected. Hence $\{a_3,b_3\}\subseteq\mathrm{supp}(u)\subseteq\{a_3,b_3,b_4\}$.

With similar arguments one shows that there is a vector $w\in X$ with $\{a_5,b_5\}\subseteq\mathrm{supp}(w)\subseteq\{a_5,b_5,b_4\}$.

If $b_4\in\mathrm{supp}(u)$ and $b_4\in\mathrm{supp}(w)$, then the vector $y:=w_{b_4}u-u_{b_4}w$ has $\{a_3,b_3,a_5,b_5\}\subseteq\mathrm{supp}(y)$, and hence $G[\mathrm{supp}(y)]$ is disconnected. Therefore $b_4\notin\mathrm{supp}(u)$ or $b_4\notin\mathrm{supp}(w)$. By symmetry we may assume that $b_4\notin\mathrm{supp}(u)$, and hence $b_4\in\mathrm{supp}(w)$ (otherwise $G[\mathrm{supp}(u+w)]$ would be disconnected.) So $\mathrm{supp}(u)=\{a_3,b_3\}$ and $\mathrm{supp}(w)=\{a_5,b_4,b_5\}$.

Let $z\in X$ be nonzero with $z_{a_1}=z_{a_3}=z_{b_2}=z_{b_4}=0$, and hence $\mathrm{supp}(z)\subseteq\{a_4,a_5,b_1,b_5\}$. If $b_1\in\mathrm{supp}(z)$, then $y:=z_{b_5}w-w_{b_5}z$ satisfies $\{b_1,b_4\}\subseteq\mathrm{supp}(y)$, and hence $G[\mathrm{supp}(y)]$ is disconnected. Hence $b_1\notin\mathrm{supp}(z)$. Let $z':=z_{a_5}w-w_{a_5}z$. Then $G[\mathrm{supp}(z'+x)]$ is disconnected, proving (6).

(7) There is no vector $x\in X$ with $\mathrm{supp}(x)=\{a_i,b_i\}$ $(i=1,\dots,5)$.

Suppose that there exists one, x say, which we may assume without loss of generality to have $\mathrm{supp}(x)=\{a_1,b_1\}$.

Let $u\in X$ be nonzero with $u_{a_1}=u_{a_4}=u_{b_3}=u_{b_5}=0$. Then $\mathrm{supp}(u)\subseteq\{a_2,a_3,b_1,b_2\}$.

Suppose $b_1\in\mathrm{supp}(u)$. Then also $b_2\in\mathrm{supp}(u)$. Let $y\in X$ be nonzero with $y_{a_2}=y_{a_5}=y_{b_1}=y_{b_4}=0$ and hence $\mathrm{supp}(y)\subseteq\{a_3,a_4,b_2,b_3\}$. If $b_2\in\mathrm{supp}(y)$ then the vector $z:=u_{b_2}y-y_{b_2}u$ has $G[\mathrm{supp}(z)]$ disconnected. Hence $b_2\notin\mathrm{supp}(y)$. But $x':=x+y$ is a vector with $\{a_1,a_3,b_1\}\subseteq\mathrm{supp}(x')\subseteq\{a_1,a_3,a_4,b_1,b_3\}$ and $G[\mathrm{supp}(x')]$ disconnected. Hence $b_1\notin\mathrm{supp}(u)$.

Exchanging the rôle of the a_i's and b_i's we can find a nonzero vector $z\in X$ with $\mathrm{supp}(z)\subseteq\{a_5,b_4,b_5\}$. But $G[\mathrm{supp}(u+z)]$ is disconnected, proving (7).

We now show that there exists a vector $x\in X$ with $\mathrm{supp}(x)=\{a_i,b_i\}$ for some i, thus obtaining a contradiction, which means that $\theta(C_5\times K_2)=4$.

Let $u\in X$ be nonzero with $u_{a_1}=u_{a_4}=u_{b_3}=u_{b_5}=0$ and let $w\in X$ be nonzero with $w_{a_1}=w_{a_4}=w_{b_2}=w_{b_3}=0$. Then $\mathrm{supp}(u)\subseteq\{a_2,a_3,b_1,b_2\}$ and $\mathrm{supp}(w)\subseteq\{a_5,b_1,b_4,b_5\}$ (as $\mathrm{supp}(w)\subseteq\{a_2,a_3\}$ cannot occur).

If $b_1\in\mathrm{supp}(u)$ and $b_1\in\mathrm{supp}(w)$ then $y:=u_{b_1}w-w_{b_1}u$ has $G[\mathrm{supp}(y)]$ disconnected. So $b_1\notin\mathrm{supp}(u)$ or $b_1\notin\mathrm{supp}(w)$. By symmetry we may assume that $b_1\notin\mathrm{supp}(u)$. (All what matters is that we find a vector

u with two vertices of $\mathrm{supp}(u)$ on one circuit of length 5 of $C_5 \times K_2$ and one vertex of $\mathrm{supp}(u)$ on the other circuit.) Then $a_3 \in \mathrm{supp}(u)$ and $b_2 \in \mathrm{supp}(u)$, and hence $\mathrm{supp}(u) = \{a_2, a_3, b_2\}$. Let $y \in X$ be nonzero with $y_{a_3} = y_{a_4} = y_{b_2} = y_{b_5} = 0$, and hence $\mathrm{supp}(y) \subseteq \{a_1, a_2, a_5, b_1\}$. If $a_2 \in \mathrm{supp}(y)$ then $z := y_{a_2} u - u_{a_2} y$ has $G[\mathrm{supp}(z)]$ disconnected. So $a_2 \notin \mathrm{supp}(y)$ and thus $\mathrm{supp}(y) = \{a_1, a_5, b_1\}$. Let $z \in X$ be nonzero with $z_{a_2} = z_{b_1} = z_{b_3} = z_{b_5} = 0$, and hence $\mathrm{supp}(z) \subseteq \{a_3, a_4, a_5, b_4\}$. If $a_5 \in \mathrm{supp}(z)$, then $z' := z_{a_5} y - y_{a_5} z$ has $G[\mathrm{supp}(z')]$ disconnected. Hence $\mathrm{supp}(z) \subseteq \{a_3, a_4, b_4\}$. If $a_3 \in \mathrm{supp}(z)$, then $z' := z_{a_3} u - u_{a_3} z$ has $G[\mathrm{supp}(z')]$ disconnected. Hence $\mathrm{supp}(z) = \{a_4, b_4\}$, which concludes the proof. ∎

Theorem 6.7. *$\nu^{=}(G) \leq 4$ if and only if G has no K_5- or $K_{2,2,2}$-minor; that is, if and only if G can be obtained by taking clique sum and subgraphs of graphs with tree-width ≤ 3 and V_8 and $C_5 \times K_2$. The same holds if $\nu^{=}(G)$ is replaced by $\theta(G)$.*

Proof. Follows from Proposition 6.4, 6.5, 6.6 and Theorem 5.6. ∎

We have shown here that $\nu^{=}(G) = \theta(G)$ if $\nu^{=}(G) \leq 4$. It is an open question if $\nu^{=}(G) = \theta(G)$ for every graph G.

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